

TWISTING 2-COCYCLES FOR THE CONSTRUCTION OF NEW NON-STANDARD QUANTUM GROUPS

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ABSTRACT. We introduce a new class of 2-cocycles defined explicitly on the generators of certain multiparameter standard quantum groups. These allow us, through the process of twisting the familiar standard quantum groups, to generate new as well as previously known examples of non-standard quantum groups. In particular we are able to construct generalisations of both the Cremmer-Gervais deformation of $SL(3)$ and the so called esoteric quantum groups of Fronsdal and Galindo in an explicit and straightforward manner.

1. INTRODUCTION

Originally there were two clearly defined types of quantum groups [1, 2, 3]. They were *single-parameter* quantisations, $U_q(\mathfrak{g})$ and $\mathbb{C}_q[G]$ respectively, of dual classical objects: the universal enveloping algebras of simple Lie algebras, $U(\mathfrak{g})$, and the coordinate rings of simple Lie groups, $\mathbb{C}[G]$. With their universal R -matrices, \mathcal{R} , these $U_q(\mathfrak{g})$ are the standard examples of quasitriangular Hopf algebras, while the $\mathbb{C}_q[G]$, together with the corresponding numerical R -matrices, are the standard examples of what we call co-quasitriangular Hopf algebras [4, 5]. It soon became apparent that there were a number of multiparameter generalisations [6, 7, 8] of these standard quantum groups and through the work of Drinfeld [9], followed by Reshetikhin [10], an interpretation emerged: all the multiparameter quantum groups corresponding to a particular standard quantum group were related, amongst themselves and with the standard quantum group, through Drinfeld's important process of twisting. In fact the original works of Drinfeld and Reshetikhin were concerned only with quasitriangular Hopf algebras, but their constructions dualise immediately to the case of co-quasitriangular Hopf algebras. Since the twists act only as similarity transformations on the so called \hat{R} -matrices [3], the different standard quantum groups corresponding to different classical Lie groups cannot be related to each other by twisting. The picture then is of a number of distinct 'twist equivalence classes'. Later, Kempf and Engeldinger [11, 12] (see also the work of Khoroshkin and Tolstoy [13]) refined Reshetikhin's work slightly and showed that there were other interesting quantum groups related, through Reshetikhin-type twists, with the standard ones.

From time to time there appeared genuinely non-standard quantum groups, usually defined in terms of non-standard numerical R -matrices. It is natural to investigate whether these define new twist equivalence classes or whether they belong to classes already defined by the standard R -matrices. We will be particularly concerned in this article with the non-standard quantum groups of Cremmer and

Gervais [14] and Fronsdal and Galindo [15, 16] which for general theoretical reasons (see Section 4) may be expected to be twist-equivalent to the standard $SL(n)$ quantum groups. However, let us be clear that *no* twist of the Reshetikhin type is suitable in these cases. As we explain later, the relevant twisting structures are counital 2-cocycles on the quantum groups. The problem then is to find the appropriate twisting 2-cocycles, defined on the standard co-quasitriangular Hopf algebra, $\mathbb{C}_q[SL(n)]$, which twist this quantum group into the Cremmer-Gervais and Fronsdal-Galindo quantum groups.

Recently, Hodges has made significant progress in this area [17], drawing on previous work of his contained in a series of important papers [18, 19, 20]. These covered many aspects of quantum group theory from a ring theoretic perspective. We should mention [19] in particular, where some remarkable aspects of the algebraic structure of Cremmer-Gervais quantum groups were revealed. In [17] Hodges starts from a particular, standard, multiparameter quantized enveloping algebra $U_p(\mathfrak{g})$. He then identifies a pair of *commuting* sub-Hopf algebras, $U_p(\mathfrak{b}_1^-)$ and $U_p(\mathfrak{b}_2^+)$, associated with certain Belavin-Drinfeld triples [21]. This gives rise to a Hopf algebra homomorphism, $\phi : U_p(\mathfrak{b}_1^-) \otimes U_p(\mathfrak{b}_2^+) \rightarrow U_p(\mathfrak{g})$, through the usual multiplication map. Attention then shifts to the dual map, $\phi^* : \mathbb{C}_p[G] \rightarrow \mathbb{C}_p[B_1^-] \otimes \mathbb{C}_p[B_2^+]$. Hodges proceeds to identify $\text{Im}(\phi^*)$, in a series of precise and subtle steps, with the image of the tensor product of a pair of ‘extended’ Borel subalgebra-like objects, between which there is a skew pairing. This skew pairing lifts to the tensor components of $\text{Im}(\phi^*)$. It is well known that such a pairing gives rise to a 2-cocycle, the quintessential example appearing in the twisting interpretation of the quantum double [5, 22], and a 2-cocycle is then induced on the quantized function algebra $\mathbb{C}_p[G]$. Hodges claims that in the particular case of $\mathfrak{sl}_3(\mathbb{C})$ the 2-cocycle coming from his construction generates the Cremmer-Gervais deformation of $\mathbb{C}[SL(3)]$. More generally, he claims that it should also be possible to reach the esoteric quantum groups of Fronsdal and Galindo [15, 16]. However, Hodges’ approach is rather technical and does not readily yield 2-cocycles defined *explicitly* on the familiar T generators of standard quantum groups. We are able to remedy this situation here.

Our approach is actually quite distinct from that of Hodges. We work entirely within the framework of co-quasitriangular Hopf algebras coming from solutions of the matrix quantum Yang-Baxter equation (QYBE). In Section 2 we recall the definition of a co-quasitriangular Hopf algebra and the basic result on twisting by 2-cocycles. A good reference for this theory, and much more besides, is the book by Majid [22], from which much of our notation is borrowed. Other good references are the paper by Larson and Towber [23] and the papers of Doi and Takeuchi [4, 5]. Majid calls co-quasitriangular Hopf algebras, ‘dual quasitriangular Hopf algebras’, but our terminology comes from [4, 5]. We go on to describe the well known class of 2-cocycles which appears as the dual of the Reshetikhin-type twists. They originate from particular solutions of the QYBE. The ‘parameterization’ twists originally considered by Reshetikhin [10] may be regarded as examples in this class, and using such a twist we have obtained a *new* 3-parameter generalized Cremmer-Gervais R -matrix, presented here, which includes as a special case the 2-parameter R -matrix considered by Hodges in [19]. Details of the derivation of this new R -matrix are given

in Appendix A. The sub-Hopf-algebra-induced twists considered by Engeldinger and Kempf [11, 12] also belong to this general class of 2-cocycles, and are recalled here.

Section 3 contains our main new results. We present there a new class of 2-cocycles which no longer emanate from solutions of the matrix QYBE. Instead, they arise from matrices satisfying a new, and remarkably simple, system of equations. A number of explicit 2-cocycles belonging to our new class are presented, along with the following results:

- It is shown *explicitly* that the *new* 3-parameter generalised Cremmer-Gervais quantum group corresponding to $GL(3)$, already given in Section 2, is obtained from a particular multiparameter standard quantum group through twisting.
- The 2-cocycle used to obtain the generalised Cremmer-Gervais deformation of $GL(3)$ is an example of a general class of *simple root* 2-cocycles which themselves belong to a more general class of *composite simple root* 2-cocycles. These 2-cocycles may all be defined on certain standard, multiparameter, deformations of $GL(n)$, and consequently generate new non-standard quantum groups.
- A 2-cocycle which can be used to twist a certain multiparameter standard deformation of $\mathbb{C}[GL(2N-1)]$ to obtain a *generalisation* of the quantum groups of Fronsdaal and Galindo is presented. For $N = 2$ this 2-cocycle is just the one used to obtain the generalised Cremmer-Gervais $GL(3)$ quantum group.

Note that the R -matrix considered by Fronsdaal and Galindo is already ‘multiparameter’, involving N parameters, but we obtain, in Appendix B, an R -matrix which, for $N > 3$, depends on $(1 + \frac{1}{2}(N-1)(N+4))$ parameters and, in the cases $N = 2$ and $N = 3$, on 3 and 7 parameters respectively. Let us also note that, as was already suggested in Hodges work [17], starting from the original standard quantum groups, $\mathbb{C}_q[SL(n)]$, we need a *combination* of the Reshetikhin-type parameterisation twists with our new twists to obtain the Fronsdaal-Galindo quantum groups.

In Section 4, we collect some information about the semi-classical objects corresponding to the R -matrices which we have been considering in this paper, namely the classical r -matrices. We also recall the background, in Drinfeld’s fundamental work, which serves as the on-going motivation in the quest for interesting twists. We end by pointing out a particular problem involved in constructing the Cremmer-Gervais quantum group corresponding to $GL(4)$, and describe an interesting phenomenon involving sub-Hopf-algebra-induced twists. Starting from the standard multiparameter quantum group, $\mathbb{C}_{q,p}[GL(4)]$, we twist, first of all, using a sub-Hopf-algebra-induced twist. The resulting quantum group may reasonably be called ‘weakly non-standard’ and can be twisted further using a 2-cocycle from our construction. The new, non-standard, R -matrix obtained through this double twist involves a pair of non-standard off-diagonal elements which could not have been added to the original R -matrix directly, but which *do* appear in the Cremmer-Gervais R -matrix for $GL(4)$.

2. CO-QUASITRIANGULAR HOPF ALGEBRAS AND RESHETIKHIN TWISTS

We begin with some basic definitions.

Definition 2.1. A bialgebra A is called *co-quasitriangular* if there exists a bilinear form σ on A , which we will call an R -form, such that

1. σ is invertible with respect to the convolution product, $*$, that is, there is another bilinear form σ^{-1} such that

$$\sigma(a_{(1)}, b_{(1)})\sigma^{-1}(a_{(2)}, b_{(2)}) = \epsilon(ab) = \sigma^{-1}(a_{(1)}, b_{(1)})\sigma(a_{(2)}, b_{(2)}), \quad (1)$$

2. $\sigma * m = m^{\text{op}} * \sigma$, i.e.

$$\sigma(a_{(1)}, b_{(1)})a_{(2)}b_{(2)} = b_{(1)}a_{(1)}\sigma(a_{(2)}, b_{(2)}), \quad (2)$$

3. $\sigma(m \otimes \text{id}) = \sigma_{13} * \sigma_{23}$, i.e.

$$\sigma(ab, c) = \sigma(a, c_{(1)})\sigma(b, c_{(2)}), \quad (3)$$

4. $\sigma(\text{id} \otimes m) = \sigma_{13} * \sigma_{12}$, i.e.

$$\sigma(a, bc) = \sigma(a_{(1)}, c)\sigma(a_{(2)}, b). \quad (4)$$

Remark 2.2. We are employing here a slightly simplified version of the Sweedler notation for coproducts: $\Delta(a) = a_{(1)} \otimes a_{(2)}$, with the summation suppressed.

Remark 2.3. It may be useful to briefly recall how we arrive at this definition. Suppose that (H, \mathcal{R}) is a quasitriangular bialgebra, with $\mathcal{R} \in H \otimes H$ the *universal R -matrix* obeying Drinfeld's familiar axioms,

$$(\Delta \otimes \text{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \quad (\text{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12}, \quad (5)$$

$$\Delta^{\text{op}}(h) = \mathcal{R} \circ \Delta(h) \circ \mathcal{R}^{-1}, \quad \forall h \in H. \quad (6)$$

In fact, we may regard \mathcal{R} as a map $k \rightarrow H \otimes H$, where k is the ground field. When formulating the dual notion of co-quasitriangular bialgebra, we then need to consider an R -form σ which is now a map $A \otimes A \rightarrow k$, where A can be thought of as dual to H . To formulate the dual axioms, involving σ instead of \mathcal{R} , we require the algebra structure on $\text{Hom}(A \otimes A, k)$. This is the convolution algebra provided by the natural tensor product coalgebra structure of $A \otimes A$. Explicitly then, let us give the details for a particular example,

$$\begin{aligned} (\sigma_{13} * \sigma_{23})(a \otimes b \otimes c) &= \sigma_{13}(a_{(1)} \otimes b_{(1)} \otimes c_{(1)})\sigma_{23}(a_{(2)} \otimes b_{(2)} \otimes c_{(2)}) \\ &= \sigma(a_{(1)}, c_{(1)})\epsilon(b_{(1)})\sigma(b_{(2)}, c_{(2)})\epsilon(a_{(2)}) \\ &= \sigma(a, c_{(1)})\sigma(b, c_{(2)}). \end{aligned}$$

For more on this process of ‘dualising’ we refer the reader to Majid’s book [22], and his paper [24].

From the definition it is readily seen that the QYBE now manifests itself as

$$\sigma_{12} * \sigma_{13} * \sigma_{23} = \sigma_{23} * \sigma_{13} * \sigma_{12}. \quad (7)$$

When A is actually a Hopf algebra, with antipode S , we call it a *co-quasitriangular Hopf algebra*. It can then be shown that S is *always* invertible, and $\sigma^{-1}(a, b) = \sigma(S(a), b)$ and $\sigma(a, b) = \sigma^{-1}(a, S(b))$.

The FRT bialgebras $A(R)$, introduced by the Leningrad school [3] and developed by Majid [25], where R is any matrix solution of the QYBE, fit into the co-quasitriangular bialgebra framework. Indeed, we define the R -form on the generators T_i^j , as

$$\sigma(T_i^s, T_j^t) = R_{ij}^{st}, \quad (8)$$

or in the useful ‘matrix notation’, as

$$\sigma(T_1, T_2) = R_{12}, \quad (9)$$

and then extend its domain of definition to the whole of $A(R)$ by setting

$$\sigma(T_1 T_2, T_3) = \sigma(T_1, T_3) \sigma(T_2, T_3) = R_{13} R_{23}, \quad (10)$$

$$\sigma(T_1, T_2 T_3) = \sigma(T_1, T_3) \sigma(T_1, T_2) = R_{13} R_{12}. \quad (11)$$

The QYBE then guarantees consistency with the product relation (2).

Remark 2.4. To remove any possible doubt about the notation being employed here, let us present (10) explicitly, in terms of the generators, as

$$\sigma(T_i^s T_j^t, T_k^r) = \sigma(T_i^s, T_k^r) \sigma(T_j^t, T_m^r) = R_{ik}^{sm} R_{jm}^{tr},$$

where the summation convention *is* being assumed.

Definition 2.5. A bilinear form χ on a bialgebra A is called a *counital 2-cocycle* on A if it is invertible in the convolution product, and

$$\chi(1, a) = \epsilon(a) = \chi(a, 1), \quad (12)$$

and

$$\chi_{12} * \chi(m \otimes \text{id}) = \chi_{23} * \chi(\text{id} \otimes m). \quad (13)$$

Remark 2.6. It is a simple matter to show that any R -form is a counital 2-cocycle. We also note that for any Hopf algebra on which we can define such a 2-cocycle, which moreover intertwines the multiplication as in (2), the antipode is necessarily invertible.

Remark 2.7. In the more familiar dual version of this definition, we consider an invertible element \mathcal{F} of $H \otimes H$. Then (12) and (13) correspond respectively to

$$(\epsilon \otimes \text{id})(\mathcal{F}) = 1 = (\text{id} \otimes \epsilon)(\mathcal{F}), \quad (14)$$

and

$$\mathcal{F}_{12}(\Delta \otimes \text{id})(\mathcal{F}) = \mathcal{F}_{23}(\text{id} \otimes \Delta)(\mathcal{F}). \quad (15)$$

An element \mathcal{F} satisfying these conditions is then called a counital 2-cocycle *for* H .

Remark 2.8. For a general discussion of cocycles for and on Hopf algebras we refer the reader to Section 2.3 of Majid’s book [22].

The property of co-quasitriangular Hopf algebras which is of particular interest to us is that, given one, we may generate others using these counital 2-cocycles. This important process of *twisting* is the dual of Drinfeld's original quasitriangular quasi-Hopf algebra twist [9], restricted to the special case of twisting *from and to* co-quasitriangular Hopf algebras. It is not difficult to dualise Drinfeld's original quasitriangular quasi-Hopf algebra axioms, and his result on twisting. This was probably first carried out explicitly by Majid [24]. We obtain the axioms for a co-quasitriangular co-quasi-Hopf algebra and on specialising the twisting result, we obtain the following important theorem.

Theorem 2.9. *Let $(A, m, \eta, \Delta, \epsilon, \sigma)$ be a co-quasitriangular bialgebra and let χ be a counital 2-cocycle on A , then there is a new co-quasitriangular bialgebra (A_χ, σ_χ) obtained by twisting the product and R -form of (A, σ) as*

$$m_\chi = \chi * m * \chi^{-1}, \quad (16)$$

$$\sigma_\chi = \chi_{21} * \sigma * \chi^{-1}. \quad (17)$$

If A is moreover a Hopf algebra with antipode S , then A_χ is also a Hopf algebra with twisted antipode given by

$$S_\chi = \lambda * S * \lambda^{-1}, \quad (18)$$

where $\lambda = \chi \circ (\text{id} \otimes S) \circ \Delta$.

For the co-quasitriangular bialgebras, $A(R)$, there is a particularly obvious way of constructing twisting 2-cocycles. Take any invertible solution F of the QYBE and define a bilinear form χ by

$$\chi(T_1, T_2) = F_{12}, \quad (19)$$

$$\chi(1, T) = \chi(T, 1) = \epsilon(T), \quad (20)$$

and

$$\chi^{-1}(T_1, T_2) = F_{12}^{-1}. \quad (21)$$

We then extend this to the whole of $A(R)$ just as we did for the R -form in equations (10) and (11), that is

$$\chi(T_1 T_2, T_3) = \chi(T_1, T_3) \chi(T_2, T_3) = F_{13} F_{23}, \quad (22)$$

$$\chi(T_1, T_2 T_3) = \chi(T_1, T_3) \chi(T_1, T_2) = F_{13} F_{12}. \quad (23)$$

However χ must respect the algebra structure already on $A(R)$ so we must also have

$$\chi(R_{12} T_1 T_2 - T_2 T_1 R_{12}, T_3) = 0$$

$$\iff R_{12} F_{13} F_{23} = F_{23} F_{13} R_{12}, \quad (24)$$

and

$$\chi(T_1, R_{23} T_2 T_3 - T_3 T_2 R_{23}) = 0$$

$$\iff R_{23} F_{13} F_{12} = F_{12} F_{13} R_{23}. \quad (25)$$

Thus any invertible solution F of the QYBE which satisfies (24) and (25) provides a 2-cocycle twist. Such a twisting system (R, F) may reasonably be called a *Reshetikhin twist* [10].

Remark 2.10. In the context of the papers [10, 11, 12], where the approach is dual to ours, an invertible element $\mathcal{F} \in H \otimes H$ is considered, where (H, \mathcal{R}) is a quasitriangular Hopf algebra. It is assumed to satisfy the QYBE,

$$\mathcal{F}_{12}\mathcal{F}_{13}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{13}\mathcal{F}_{12}, \quad (26)$$

and the relations

$$(\Delta \otimes \text{id})(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{23}, \quad (27)$$

and

$$(\text{id} \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{13}\mathcal{F}_{12}, \quad (28)$$

which correspond respectively to equations (22) and (23). This \mathcal{F} is then a 2-cocycle for H in the sense of Remark 2.7, and twists the *comultiplication*, *universal R -matrix* and antipode as

$$\Delta_{\mathcal{F}}(h) = \mathcal{F}\Delta(h)\mathcal{F}^{-1}, \quad \forall h \in H, \quad (29)$$

$$\mathcal{R}_{\mathcal{F}} = \mathcal{F}_{21}\mathcal{R}\mathcal{F}^{-1}, \quad (30)$$

and

$$S_{\mathcal{F}}(h) = vS(h)v^{-1}, \quad \forall h \in H, \quad (31)$$

where $v = m \circ (\text{id} \otimes S)(\mathcal{F})$. This is actually a slight generalisation of the presentation of Reshetikhin [10], due to Kempf and Engeldinger [11, 12].

We will be particularly interested in the situation pertaining when we take R to be the standard $SL(n)$ type R -matrix given by,

$$(R_S)_{ij}^{st} = \begin{cases} q & i = j = s = t, \\ 1 & i = s \neq j = t, \\ (q - q^{-1}) & i = t < j = s. \end{cases} \quad (32)$$

In this case, with the identification of the central quantum determinant, $A(R)$ becomes a Hopf algebra. Indeed it is the standard quantization of the coordinate ring of the Lie group $SL(n)$, denoted $\mathbb{C}_q[SL(n)]$.

Let us also present here the expressions, in our notation, for the Cremmer-Gervais $SL(n)$ R -matrix and the Fronsdaal-Galindo $GL(2N - 1)$ R -matrix. First, the Cremmer-Gervais R -matrix will be taken to be

$$(R_{CG})_{ij}^{st} = \begin{cases} q & i = j = s = t, \\ qq^{-2(j-s)/n} & i = s < j = t, \\ q^{-1}q^{-2(j-s)/n} & i = s > j = t, \\ (q - q^{-1}) & i = t < j = s, \\ (q - q^{-1})q^{-2(j-s)/n} & i < s < j, \text{ and } t = i + j - s, \\ -(q - q^{-1})q^{-2(j-s)/n} & j < s < i, \text{ and } t = i + j - s. \end{cases} \quad (33)$$

If the R -matrix which appears as equation (46) in the original paper of Cremmer and Gervais [14] is denoted \tilde{R} then $R_{CG} = \tilde{R}_{21}$ (with e^{-ih} there replaced by q here). $A(R_{CG})$ again becomes a Hopf algebra, with the identification of the central quantum determinant found in [19], and will then be denoted $\mathbb{C}_{CG,q}[SL(n)]$.

The Fronsdaal-Galindo R -matrix will be considered in the next section. However we will present it here so that the reader might easily compare it with the Cremmer-Gervais R -matrix. Thus, we take the Fronsdaal-Galindo R -matrix to be

$$(R_{FG})_{ij}^{st} = \begin{cases} q & i = j = s = t, \\ q & i = s = 2N - j, j = t, 0 < j < N, \\ q^{-1} & i = s, j = t = 2N - i, 0 < i < N, \\ 1 & i = s, j = t, i \neq j, i + j \neq 2N, \\ q - q^{-1} & i = t < j = s, \\ q\kappa_i & 0 < i < N, j = 2N - i, s = t = N, \\ q\tilde{\kappa}_j & 0 < j < N, i = 2N - j, s = t = N, \\ q^{-1}\tilde{\xi}_{is} & 0 < i < s < N, j = 2N - i, t = 2N - s, \\ q\tilde{\xi}_{jt} & 0 < j < t < N, i = 2N - j, s = 2N - t, \end{cases} \quad (34)$$

where

$$\tilde{\kappa}_i = -q^{2(N-i)}\kappa_i, \quad (35)$$

$$\tilde{\xi}_{ij} = (1 - q^{-2})q^{2(j-i)}(\kappa_i/\kappa_j), \quad (36)$$

$$\xi_{ij} = (1 - q^2)(\kappa_i/\kappa_j). \quad (37)$$

Clearly, R_{FG} depends on N parameters — q together with κ_i , $0 < i < N$. In this case, if the R -matrix given in section 5 of the paper [15] (with the q there replaced by q^{-1}) is denoted \tilde{R} , then $R_{FG} = q\tilde{R}_{21}$. It will be shown that this R -matrix is related via twisting to R_S . Thus we can say that $A(R_{FG})$ may also be taken to be a Hopf algebra, which is more than is claimed in [15, 16]. We will denote this quantum group by $\mathbb{C}_{FG,q}[GL(2N - 1)]$.

Example 2.11. For the standard deformation $\mathbb{C}_q[SL(n)]$, defined by the R -form $\sigma_S(T_i^s, T_j^t) = (R_S)_{ij}^{st}$, we define the 2-cocycle χ by

$$\chi(T_i^s, T_j^t) = F_{ij}^{st} = f_{ij}\delta_i^s\delta_j^t, \quad (38)$$

extended to the whole of $\mathbb{C}_q[SL(n)]$ by (22) and (23). Note that the summation convention is not being assumed here and indeed will not be assumed anywhere, unless stated otherwise. We find that the conditions (24) and (25) impose no restrictions on the f_{ij} s. The new, twisted R -form then coming from (17) is

$$\sigma_{S,p}(T_i^s, T_j^t) = (R_{S,p})_{ij}^{st}, \quad (39)$$

where $R_{S,p}$ is the familiar $(1 + \binom{n}{2})$ -parameter standard R -matrix given by

$$(R_{S,p})_{ij}^{st} = \begin{cases} q & i = j = s = t, \\ p_{ij} & i = s \neq j = t, \\ (q - q^{-1}) & i = t < j = s, \end{cases} \quad (40)$$

with $p_{ij} = f_{ji}f_{ij}^{-1} = p_{ji}^{-1}$ for $i < j$. The multiparameter Hopf algebra so defined will be denoted $\mathbb{C}_{q,p}[GL(n)]$, and was first constructed in this way by Kempf in [11] (see also the paper by Schirmacher [26]). We will often take $p_{ii} = q$ in what follows.

Example 2.12. For the 1-parameter Cremmer-Gervais deformation $\mathbb{C}_{CG,q}[SL(n)]$ the situation is more interesting. We again define a 2-cocycle χ in terms of a diagonal matrix $F_{ij}^{st} = f_{ij}\delta_i^s\delta_j^t$. However now the compatibility conditions (24) and (25) *do* impose restrictions on the f_{ij} s. The number of independent parameters appearing in the twisted Hopf algebra $\mathbb{C}_{CG,q,p}[GL(n)]$ is then determined by the number of independent combinations of the f_{ij} s which appear in $R_{CG,p} = F_{21}R_{CG}F^{-1}$. As demonstrated in Appendix A, we are left with just three independent parameters — q together with a new pair, p and λ . Explicitly, the 3-parameter generalised Cremmer-Gervais R -matrix is given by

$$(R_{CG,p})_{ij}^{st} = \begin{cases} q & i = j = s = t, \\ p^{j-s}q & i = s < j = t, \\ p^{j-s}q^{-1} & i = s > j = t, \\ (q - q^{-1}) & i = t < j = s, \\ p^{j-s}\lambda^{st-ij}(q - q^{-1}) & i < s < j, \text{ and } t = i + j - s, \\ -p^{j-s}\lambda^{st-ij}(q - q^{-1}) & j < s < i, \text{ and } t = i + j - s. \end{cases} \quad (41)$$

We refer the reader to Appendix A for the proof of this result.

Example 2.13. Another type of Reshetikhin twist, is the sub-Hopf-algebra-induced twist, studied in particular by Engeldinger and Kempf [12]. An example of such a twist is given by defining a 2-cocycle on $\mathbb{C}_{q,p}[GL(n)]$ as

$$\chi(T_i^s, T_j^t) = \begin{cases} f_{ij} & i = s, j = t, \\ q^{-1}(q - q^{-1})f_{\eta\eta} & i = t = \eta, j = s = \eta + 1, \end{cases} \quad (42)$$

with the following restrictions on the f_{ij} s to ensure that all the conditions of the twisting system are satisfied,

$$f_{\eta\eta} = f_{\eta+1,\eta+1}, \quad (43)$$

$$f_{\eta,\eta+1} = q^{-1}p_{\eta,\eta+1}f_{\eta\eta}, \quad (44)$$

$$f_{\eta+1,\eta} = q^{-1}p_{\eta+1,\eta}f_{\eta\eta}, \quad (45)$$

$$f_{i,\eta+1} = p_{i,\eta+1}p_{\eta,i}f_{i\eta}, \quad i \neq \eta, \eta + 1, \quad (46)$$

$$f_{\eta+1,i} = p_{\eta+1,i}p_{i,\eta}f_{\eta i}, \quad i \neq \eta, \eta + 1. \quad (47)$$

The new R -form is then given by the R -matrix

$$(R_{EK})_{ij}^{st} = \begin{cases} q & i = j = s = t, \\ \tilde{p}_{ij} & i = s \neq j = t, \\ q - q^{-1} & i = t < j = s, \\ -(q - q^{-1}) & i = t = \eta, j = s = \eta + 1, \\ q - q^{-1} & i = t = \eta + 1, j = s = \eta, \end{cases} \quad (48)$$

where $\tilde{p}_{ij} = p_{ij}f_{ji}f_{ij}^{-1}$. There is no change in the number of independent parameters. The twist quoted here actually corresponds to an embedding of $U_q(\mathfrak{gl}_2(\mathbb{C}))$ in $U_q(\mathfrak{gl}_n(\mathbb{C}))$. There are many others, and we refer the reader to [12] for details.

3. A NEW CLASS OF TWISTING 2-COCYCLES

Our major results all appear as particular examples of a new twisting system, quite distinct from that of Reshetikhin, described in the following theorem.

Theorem 3.1. *Suppose $A(R)$ is any FRT bialgebra, defined in terms of an $n \times n$ R -matrix R . To any $n \times n$ matrix F which satisfies the following conditions,*

$$F_{12}F_{23} = F_{23}F_{12}, \quad (49)$$

$$R_{12}F_{23}F_{13} = F_{13}F_{23}R_{12}, \quad (50)$$

$$R_{23}F_{12}F_{13} = F_{13}F_{12}R_{23}, \quad (51)$$

there corresponds a counital 2-cocycle χ defined on $A(R)$. It is given on the generators of $A(R)$ by

$$\chi(1, T) = \chi(T, 1) = \epsilon(T), \quad (52)$$

$$\chi(T_1, T_2) = F_{12}, \quad (53)$$

and extended to the whole algebra as

$$\chi(T_1T_2, T_3) = \chi(T_2, T_3)\chi(T_1, T_3) = F_{23}F_{13}, \quad (54)$$

$$\chi(T_1, T_2T_3) = \chi(T_1, T_2)\chi(T_1, T_3) = F_{12}F_{13}. \quad (55)$$

Proof. The fact that χ is consistent with the underlying algebraic structure of $A(R)$ follows from (50) and (51), while the defining condition (13) follows easily on using (49) together with (54), (55) and the fact that $F_{ij}F_{kl} = F_{kl}F_{ij}$ whenever i, j, k and l are mutually distinct. \square

Remark 3.2. There is of course a dual result to this, which applies to any quasitriangular Hopf algebra (H, \mathcal{R}) : Given an invertible element $\mathcal{F} \in H \otimes H$ satisfying

$$\mathcal{F}_{12}\mathcal{F}_{23} = \mathcal{F}_{23}\mathcal{F}_{12}, \quad (56)$$

together with

$$(\Delta \otimes \text{id})(\mathcal{F}) = \mathcal{F}_{23}\mathcal{F}_{13}, \quad (57)$$

and

$$(\text{id} \otimes \Delta)(\mathcal{F}) = \mathcal{F}_{12}\mathcal{F}_{13}, \quad (58)$$

then $(H, \mathcal{R}_{\mathcal{F}})$ is a new quasitriangular Hopf algebra, with the coproduct, universal R -matrix and antipode twisted as in (29), (30) and (31) respectively.

Some of the general features of twists coming from this construction will be explicated in the following example.

Example 3.3. Let us take as our initial object, the multiparameter standard quantum group $\mathbb{C}_{q,p}[GL(3)]$, and consider the possibility of defining on it a 2-cocycle χ defined on the generators as

$$\chi(T_i^s, T_j^t) = F_{ij}^{st} = \begin{cases} f_{ij} & i = s, j = t, \\ \mu & i = 1, j = 3, s = t = 2. \end{cases} \quad (59)$$

For F to satisfy (49) we need $f_{i1} = f_{i2}$ and $f_{2i} = f_{3i}$ for all $i = 1, \dots, 3$. For χ to be compatible with the algebra structure of $\mathbb{C}_{q,p}[GL(3)]$ we need (50) and (51) to be satisfied, which further requires $p_{i2}f_{i3} = p_{i1}f_{i2}$ and $p_{3i}f_{2i} = p_{2i}f_{1i}$ where $p_{ii} = q$, for $i = 1, \dots, 3$. For generic p_{ij} these equations have no solution. However, giving up a degree of freedom from the parameter space of $\mathbb{C}_{q,p}[GL(3)]$ by setting $p_{13} = qp_{12}p_{23}$, they can be solved. As a matrix, F is then given by

$$F = \begin{pmatrix} q^{-1}p_{32}f & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & q^{-1}p_{32}f & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & p_{21}p_{32}f & 0 & \mu & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & f & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & f & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & q^{-1}p_{21}f & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & f & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & f & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1}p_{21}f \end{pmatrix}, \quad (60)$$

where $f = f_{22}$. The R -matrix of $\mathbb{C}_{q,p}[SL(3)]$ with $p_{13} = qp_{12}p_{23}$ then twists to R_χ , where

$$R_\chi = \begin{pmatrix} q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & qp & 0 & q - q^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & qp^2 & 0 & -p^2qf^{-1}\mu & 0 & q - q^{-1} & 0 & 0 \\ 0 & 0 & 0 & q^{-1}p^{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & qp & 0 & q - q^{-1} & 0 \\ 0 & 0 & 0 & 0 & qf^{-1}\mu & 0 & q^{-1}p^{-2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1}p^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q \end{pmatrix}, \quad (61)$$

with $p = p_{12}p_{23}$. It is clear that on choosing $f = -p\lambda^{-1}$ and $\mu = q^{-1}(q - q^{-1})$, we have obtained precisely the R -matrix $R_{CG,p}$ for $n = 3$.

The 2-cocycle here is an example of a general class of *simple root 2-cocycles* which are defined on $\mathbb{C}_{q,p}[GL(n)]$ for any pair of integers (k, l) such that $0 < k < l < n$ by

$$\chi(T_i^s, T_j^t) = F_{ij}^{st} = \begin{cases} f_{ij} & i = s, j = t, \\ \mu & i = k, j = l + 1, s = k + 1, t = l, \end{cases} \quad (62)$$

with the constraints

$$f_{i,k} = f_{i,k+1}, \quad f_{l,i} = f_{l+1,i}, \quad (63)$$

and

$$p_{i,k}f_{i,l} = p_{i,k+1}f_{i,l+1}, \quad (64)$$

$$p_{l,i}f_{k,i} = p_{l+1,i}f_{k+1,i}, \quad (65)$$

for all $i = 1, \dots, n$. The name comes from the fact that these twists add non-zero elements to the R -matrix at points corresponding to the non-zero elements of the matrices $\Gamma(e_{\alpha_k}) \otimes \Gamma(e_{-\alpha_l})$, and $\Gamma(e_{-\alpha_l}) \otimes \Gamma(e_{\alpha_k})$, where the e_{α_k} are the basis elements corresponding to the simple roots of the Lie algebra $\mathfrak{gl}_n(\mathbb{C})$, and Γ is the first fundamental representation. These simple root 2-cocycles may be combined in more general *composite simple root 2-cocycles* defined on $\mathbb{C}_{q,p}[GL(n)]$ for each $0 < k < n$, by

$$\chi(T_i^s, T_j^t) = F_{ij}^{st} = \begin{cases} f_{ij} & i = s, j = t, \\ \mu_m & i = k, j = m+1, s = k+1, t = m. \end{cases} \quad (66)$$

With the constraints as before, and m now taking all possible values such that $k < m < n$, this twist imparts a whole series of non-standard off-diagonal elements to the R -matrix. Demonstration of the truth of these statements involves a straightforward verification of the conditions (49), (50) and (51).

While working on the universal T -matrix, Fronsdaal and Galindo [15, 16] found an interesting non-standard deformation of $\mathbb{C}[GL(2N-1)]$, which we shall denote by $\mathbb{C}_{FG,q}[GL(2N-1)]$, and which they called ‘esoteric’. We have already introduced their R -matrix in (34). For $N = 2$ this is precisely the generalised Cremmer-Gervais quantum group $\mathbb{C}_{CG,q,p}[GL(3)]$ with $p = q^{-1}$ and $\lambda = q^2(\kappa_1/(q - q^{-1}))$. However for $N > 2$, their quantum groups do not coincide with those of the Cremmer-Gervais series (c.f. added note in [16]). In fact, in a sense which we will make more precise in the next section, the quantum groups $\mathbb{C}_{FG,q}[GL(2N-1)]$ are ‘not as non-standard’ as those of Cremmer and Gervais. As we shall discuss later, the general Cremmer-Gervais quantum group does not seem to be a twisting of a standard quantum group by a 2-cocycle of the type we are considering here. However the quantum groups of Fronsdaal and Galindo *are* obtained from standard-type quantum groups through a 2-cocycle which fits into our general scheme, and is presented in the following proposition.

Proposition 3.4. *On the quantum group $\mathbb{C}_{q,p}[GL(2N-1)]$, with the parameters constrained according to*

$$p_{ji'} = qp_j N p_{Ni'}, \quad \frac{p_{ij}}{p_i N p_{Nj}} = \frac{p_{i'j'}}{p_{i'N} p_{Nj'}}, \quad (67)$$

for all $0 < i, j < N$ and where $i' = 2N - i$, there is a 2-cocycle χ_{FG} defined as

$$\chi_{FG}(T_i^s, T_j^t) = F_{ij}^{st} = \begin{cases} f_{ij} & i = s, j = t, \\ \mu_k & i = k, j = k', s = N, t = N, \\ \lambda_{kl} & i = k, j = k', s = l, t = l', \end{cases} \quad (68)$$

where $0 < k < l < N$. All the f_{ij} are given in terms of f_{NN} according to

$$f_{ij} = \begin{cases} q^{-1}p_{i'N}f_{NN} & 0 < i, j \leq N, \\ p_{i'j}p_{jj'}f_{NN} & 0 < i \leq N < j < 2N, \\ f_{NN} & 0 < j \leq N < i < 2N, \\ q^{-1}p_{Nj'}f_{NN} & N < i, j < 2N. \end{cases} \quad (69)$$

The λ s are determined in terms of the μ s by

$$\lambda_{ij} = p_{j'j}f_{NN}(q - q^{-1})(\mu_i/\mu_j), \quad (70)$$

for all $0 < i < j < N$.

Proof. This result is obtained by applying conditions (49), (50) and (51), with $R = R_{S,p}$, to (68). \square

Remark 3.5. It is not difficult to establish that the number of parameters left in $\mathbb{C}_{q,p}[GL(2N-1)]$ after imposing the conditions (67), is $(1 + \frac{1}{2}(N-1)(N+2))$. This is just the number of independent p_{ij} s.

Using the 2-cocycle (68) we in fact obtain an R -matrix more general than that of Fronsdaal and Galindo. Details are given in Appendix B, where we obtain this *multiparameter generalised Fronsdaal-Galindo R -matrix* and demonstrate explicitly that their original R -matrix (34) is a special case of the new generalised R -matrix.

4. THE CREMMER-GERVAIS PROBLEM FOR $GL(4)$ AND BEYOND

In the semiclassical theory of quasitriangular Lie bialgebras associated with Lie algebras \mathfrak{g} , and their corresponding Poisson Lie groups (see for example the treatment in the book by Chari and Pressley [27]), the fundamental role is played by the classical r -matrix, $r \in \mathfrak{g} \otimes \mathfrak{g}$, which completely specifies the Lie bialgebra. In the case of complex, finite dimensional, simple Lie algebras, there is a complete classification of all such r -matrices, due to Belavin and Drinfeld [21, 27], in terms of ‘admissible’ or ‘Belavin-Drinfeld’ triples, (Π_1, Π_0, τ) , where Π is the set of simple roots of \mathfrak{g} , $\Pi_1, \Pi_0 \subset \Pi$ and $\tau : \Pi_1 \rightarrow \Pi_0$ is a bijection. Considering in particular the situation for $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, we can distinguish three cases of interest to us. In the standard, or Drinfeld-Jimbo case, the Belavin-Drinfeld triple has Π_1 and Π_0 both empty and the corresponding r -matrix, r_S , coincides with the semiclassical limit of the universal R -matrix, \mathcal{R}_S , of the familiar quasitriangular quantized universal enveloping algebra, $U_h(\mathfrak{sl}_n(\mathbb{C}))$. Another r -matrix, this time for $\mathfrak{sl}_{2N-1}(\mathbb{C})$, has [17] $\Pi_1 = \{\alpha_1, \alpha_2, \dots, \alpha_{N-1}\}$ and $\Pi_0 = \{\alpha_N, \alpha_{N+1}, \dots, \alpha_{2(N-1)}\}$, where $\alpha_1, \dots, \alpha_{2(N-1)}$ are the simple roots of $\mathfrak{sl}_{2N-1}(\mathbb{C})$. When considered in the first fundamental representation of $\mathfrak{sl}_{2N-1}(\mathbb{C})$, this can be seen to correspond to the semiclassical limit of a Fronsdaal-Galindo type R -matrix (34), and so will be denoted r_{FG} . Finally, we have a r -matrix for $\mathfrak{sl}_n(\mathbb{C})$, in which Π_1 and Π_0 are as full as possible [28], with $\Pi_1 = \{\alpha_1, \alpha_2, \dots, \alpha_{n-2}\}$ and $\Pi_0 = \{\alpha_2, \alpha_{n+1}, \dots, \alpha_{n-1}\}$, where $\alpha_1, \dots, \alpha_{n-1}$ are the simple roots of $\mathfrak{sl}_n(\mathbb{C})$. This time, when viewed in the first fundamental representation of $\mathfrak{sl}_n(\mathbb{C})$, we find a correspondence with a Cremmer-Gervais type R -matrix (33), and so we will write this r -matrix as r_{CG} . Let us note, that in each

of these cases, the element t defined as $t = r_{12} + r_{21}$ is identical, and is in fact the Casimir element of $\mathfrak{sl}_n(\mathbb{C}) \otimes \mathfrak{sl}_n(\mathbb{C})$.

In a series of fundamental works [9, 29, 30], Drinfeld proved that given any Lie algebra, \mathfrak{g} , together with a symmetric \mathfrak{g} -invariant element, t , there exists a quantization of the universal enveloping algebra, $U(\mathfrak{g})$, as a *quasitriangular quasi-Hopf quantized universal enveloping algebra*, $(U(\mathfrak{g})[[\hbar]], \Phi, e^{\hbar t/2})$, and that this quantization is *unique up to twisting*. An immediate consequence of this result is that the standard quantization, $(U_h(\mathfrak{sl}_n(\mathbb{C})), \mathcal{R}_S)$ of $U(\mathfrak{sl}_n(\mathbb{C}))$, is twist equivalent, as a *quasitriangular quasi-Hopf algebra*, to the ‘universal’ quantization $(U(\mathfrak{sl}_n(\mathbb{C}))[[\hbar]], \Phi, e^{\hbar t/2})$. Subsequently [31], Drinfeld formulated a number of unsolved problems in quantum group theory. Among these, was the question of whether *every* finite dimensional Lie bialgebra admits a quantization as a quantized universal enveloping algebra. This was recently answered, in the affirmative, by Etingof and Kazhdan [32]. Though their result did not provide an explicit construction, it does tell us that in addition to the well known Drinfeld-Jimbo quasitriangular quantized universal enveloping algebra, $(U_h(\mathfrak{sl}_n(\mathbb{C})), \mathcal{R}_S)$, we must assume the *existence* of quasitriangular Fronsda-Galindo and Cremmer-Gervais quantized universal enveloping algebras, with corresponding universal R -matrices \mathcal{R}_{FG} and \mathcal{R}_{CG} respectively. Moreover, by Drinfeld’s earlier result, we know that these quantized universal enveloping algebras must be twist equivalent as *quasitriangular Hopf algebras*, to $(U_h(\mathfrak{sl}_n(\mathbb{C})), \mathcal{R}_S)$. In particular, the universal R -matrices, \mathcal{R}_S , \mathcal{R}_{FG} , and \mathcal{R}_{CG} , must each be related to each other by twisting in the style of equation (30).

It is reasonable, we believe, to work under the motivating assumption that the matrices R_{FG} and R_{CG} , which have been considered in this paper, correspond to the, as yet unknown, universal R -matrices, \mathcal{R}_{FG} and \mathcal{R}_{CG} , in the first fundamental representation. In this case, the theory we have just outlined implies that in the dual world of co-quasitriangular Hopf algebras, there should exist 2-cocycles for the construction of the Fronsda-Galindo quantum groups and the Cremmer-Gervais quantum groups from the standard quantum groups. Some support for the assumption has been provided in this paper, with the explicit construction of a twisting 2-cocycle for the construction of the Fronsda-Galindo quantum groups, and the Cremmer-Gervais deformation of $GL(3)$. However the problem for the Cremmer-Gervais deformations of $GL(n)$ for $n > 3$ remains open.

The pair of non-standard off-diagonal elements which appear in the Cremmer-Gervais R -matrix for $GL(3)$ ‘correspond’ in the sense described above, to the element $e_{\alpha_1} \wedge e_{-\alpha_2}$ of the corresponding classical r -matrix, r_{CG} . As we have seen, our twisting construction has no problem dealing with this case. However, for $GL(4)$ and beyond, the Cremmer-Gervais R -matrix involves an increasing number of non-simple root combinations — more than appear in the Fronsda-Galindo R -matrix, and our construction does not appear to be able to deal with this circumstance. In particular, in the Cremmer-Gervais R -matrix for $GL(4)$, there are pairs of non-standard matrix elements corresponding to the r -matrix elements $e_{\alpha_1} \wedge e_{-\alpha_2}$, $e_{\alpha_1} \wedge e_{-\alpha_3}$, $e_{\alpha_2} \wedge e_{-\alpha_3}$ and $e_{\alpha_1+\alpha_2} \wedge e_{-(\alpha_2+\alpha_3)}$. The last term, in particular, causes problems. We finish by explaining how a *new* non-standard $GL(4)$ R -matrix may be obtained, which contains a pair of matrix elements corresponding to this term.

Starting from the standard quantum group $\mathbb{C}_{q,p}[GL(4)]$, and twisting first using a 2-cocycle of the kind in (42), we obtain a new quantum group given in terms of the R -matrix,

$$(R_{EK})_{ij}^{st} = \begin{cases} q & i = j = s = t, \\ \tilde{p}_{ij} & i = s \neq j = t, \\ q - q^{-1} & i = t < j = s, \\ -(q - q^{-1}) & i = t = 2, j = s = 3, \\ q - q^{-1} & i = t = 3, j = s = 2. \end{cases} \quad (71)$$

This quantum group is now amenable to a twist by one of our new 2-cocycles, given by

$$\chi(T_i^s, T_j^t) = F_{ij}^{st} = \begin{cases} f_{ij} & i = s, j = t, \\ \lambda & i = 1, j = 4, s = 3, t = 2, \end{cases} \quad (72)$$

with the constraints,

$$f_{i1} = f_{i3}, \quad f_{2i} = f_{4i}, \quad (73)$$

$$\tilde{p}_{i1}f_{i2} = \tilde{p}_{i3}f_{i4}, \quad \tilde{p}_{4i}f_{3i} = \tilde{p}_{2i}f_{1i}, \quad (74)$$

for $i = 1, \dots, 4$. Note that the this 2-cocycle could not have been defined on the original standard R -matrix. The R -matrix for this *new* non-standard quantum group may now be written as

$$(R_{NS})_{ij}^{st} = \begin{cases} q & i = j = s = t, \\ \gamma_{ij} & i = s \neq j = t, \\ q - q^{-1} & i = t < j = s, \\ -(q - q^{-1}) & i = t = 2, j = s = 3, \\ q - q^{-1} & i = t = 3, j = s = 2, \\ \gamma_{14}\varrho & i = 1, j = 4, s = 3, t = 2, \\ -q\gamma_{23}\varrho & i = 4, j = 1, s = 2, t = 3, \end{cases} \quad (75)$$

where

$$\gamma_{ij} = \tilde{p}_{ij}f_{ji}f_{ij}^{-1}, \quad \varrho = -\lambda f_{14}^{-1}f_{32}^{-1}, \quad (76)$$

and

$$\gamma_{12}\gamma_{23} = q\gamma_{24}, \quad \gamma_{24}\gamma_{34} = q\gamma_{14}. \quad (77)$$

This new R -matrix depends on 6 parameters. It might be interesting to investigate such double twists further.

APPENDIX A. THE 3-PARAMETER GENERALISED CREMMER-GERVAIS R -MATRIX

We give here the derivation of the result quoted in Example 2.12. We proceed in two stages, obtaining the required result by demonstrating that the R -matrix (41) is a twist of the R -matrix (33).

1. As explained in Section 2, as any diagonal matrix $F_{ij}^{kl} = f_{ij}\delta_i^k\delta_j^l$ is a solution of the QYBE, we can define a twisting 2-cocycle χ in terms of it as $\chi(T_1, T_2) = F_{12}$ as

long as the compatibility conditions (24) and (25) are satisfied. In terms of matrix components, these conditions become

$$R_{ij}^{st} f_{s\alpha} f_{t\alpha} = f_{j\alpha} f_{i\alpha} R_{ij}^{st}, \quad (\text{A1})$$

and

$$R_{ij}^{st} f_{\alpha t} f_{\alpha s} = f_{\alpha i} f_{\alpha j} R_{ij}^{st}, \quad (\text{A2})$$

$i, j, \alpha, s, t = 1, \dots, n$, $n \geq 3$. Thus the non-zero elements of the R -matrix R_{CG} determine the constraints on the elements of F . It is not difficult to see that the only non-trivial relations which we get are

$$f_{i\alpha} f_{j\alpha} = f_{s\alpha} f_{t\alpha} \quad i < s < j, \quad t = i + j - s, \quad (\text{A3})$$

and

$$f_{\alpha i} f_{\alpha j} = f_{\alpha s} f_{\alpha t} \quad i < s < j, \quad t = i + j - s, \quad (\text{A4})$$

$i, j, \alpha, s, t = 1, \dots, n$, $n \geq 3$. We will now prove the following lemma.

Lemma. *The system of equations (A3) and (A4), in n^2 unknowns, has a solution space completely described in terms of four unknowns x, y, z, w say, as*

$$f_{ij} = x^{(i-2)(j-2)} y^{-(i-2)(j-1)} z^{-(i-1)(j-2)} w^{(i-1)(j-1)}, \quad (\text{A5})$$

$i, j = 1, \dots, n$.

Proof. We use induction. Consider the simplest case, $n = 3$, so that we have $i = 1$, $s = t = 2$, $j = 3$ and there are six equations

$$f_{11} f_{31} = f_{21}^2, \quad (\text{A6})$$

$$f_{12} f_{32} = f_{22}^2, \quad (\text{A7})$$

$$f_{13} f_{33} = f_{23}^2, \quad (\text{A8})$$

$$f_{11} f_{13} = f_{12}^2, \quad (\text{A9})$$

$$f_{21} f_{23} = f_{22}^2, \quad (\text{A10})$$

$$f_{31} f_{33} = f_{32}^2. \quad (\text{A11})$$

Only five of these are independent, e.g. combining (A6), (A7), (A8), (A9) and (A10) yields (A11), so the solution space will be in terms of four unknowns. Choosing these to be $f_{11} = x$, $f_{12} = y$, $f_{21} = z$ and $f_{22} = w$, we find

$$\|f_{ij}\| = \begin{pmatrix} x & y & x^{-1}y^2 \\ z & w & z^{-1}w^2 \\ x^{-1}z^2 & y^{-1}w^2 & xy^{-2}z^{-2}w^4 \end{pmatrix}, \quad (\text{A12})$$

which verifies (A5) for $n = 3$. Now suppose that the solution space of the system of equations (A3) and (A4) for $n = k$, $k \geq 3$ is completely specified by (A5), and consider $n = k + 1$. Notice that we still have all the equations we had for $n = k$ so (A5) holds for $i, j = 1, \dots, k$. We need to check that the new equations appearing consistently specify $f_{\alpha, k+1}$ and $f_{k+1, \alpha}$ according to (A5) for $\alpha = 1, \dots, k + 1$.

From (A4), for $\alpha = 1, \dots, k$,

$$\begin{aligned}
f_{\alpha,k+1} &= f_{\alpha i}^{-1} f_{\alpha s} f_{\alpha t}, \\
&= x^{-(\alpha-2)(i-2)+(\alpha-2)(s-2)+(\alpha-2)(t-2)} \\
&\quad \times y^{(\alpha-2)(i-1)-(\alpha-2)(s-1)-(\alpha-2)(t-1)} \\
&\quad \times z^{(\alpha-1)(i-2)-(\alpha-1)(s-2)-(\alpha-1)(t-2)} \\
&\quad \times w^{-(\alpha-1)(i-1)+(\alpha-1)(s-1)+(\alpha-1)(t-1)}, \\
&= x^{(\alpha-2)(-i+s+t-2)} y^{(\alpha-2)(i-s-t+1)} z^{(\alpha-1)(i-s-t+2)} w^{(\alpha-1)(-i+s+t-1)}, \\
&= x^{(\alpha-2)(k+1-2)} y^{-(\alpha-2)(k+1-1)} z^{-(\alpha-1)(k+1-2)} w^{(\alpha-1)(k+1-1)},
\end{aligned}$$

as required. Invoking the obvious symmetry between (A3) and (A4) we deduce the equivalent result from (A3) for $f_{k+1,\alpha}$, $\alpha = 1, \dots, k$. Replacing α by $k+1$ in the above computation yields the correct result for $f_{k+1,k+1}$. The consistency of these solutions still needs to be checked, but follows from the following. Take $f_{\alpha,k+1} = f_{\alpha s} f_{\alpha t} f_{\alpha i}^{-1}$ from (A4), and consider (A3) with $\alpha = k+1$, i.e.

$$f_{i',k+1} f_{j',k+1} = f_{s',k+1} f_{t',k+1},$$

where $i' < s' < j'$, and $t' = i' + j' - s'$, $i', s', t', j' = 1, \dots, k+1$. Then

$$\begin{aligned}
\text{LHS} &= f_{i's} f_{i't} f_{i'i}^{-1} f_{j's} f_{j't} f_{j'i}^{-1} \\
&= f_{i's} f_{i't} f_{j'i} f_{s'i}^{-1} f_{t'i}^{-1} f_{j's} f_{j't} f_{j'i}^{-1} \\
&= f_{s's} f_{t's} f_{s't} f_{t't} f_{s'i}^{-1} f_{t'i}^{-1} \\
&= f_{s',k+1} f_{t',k+1} \\
&= \text{RHS}
\end{aligned}$$

□

2. We must now consider what combinations of the f_{ij} s actually take part in the twisting. Thus, we consider the matrix $R_{CG,p} = F_{21} R_{CG} F^{-1}$, whose components are given by $(R_{CG,p})_{ij}^{st} = f_{ji} (R_{CG})_{ij}^{st} f_{st}^{-1}$. Explicitly,

$$(R_{CG,p})_{ij}^{st} = \begin{cases} q & i = j = s = t, \\ f_{ji} f_{ij}^{-1} q q^{-2(j-s)/n} & i = s < j = t, \\ f_{ji} f_{ij}^{-1} q^{-1} q^{-2(j-s)/n} & i = s > j = t, \\ (q - q^{-1}) & i = t < j = s, \\ f_{ji} f_{st}^{-1} (q - q^{-1}) q^{-2(j-s)/n} & i < s < j, \text{ and } t = i + j - s, \\ -f_{ji} f_{st}^{-1} (q - q^{-1}) q^{-2(j-s)/n} & j < s < i, \text{ and } t = i + j - s, \end{cases} \quad (\text{A13})$$

and we are led to define

$$q_{ij} = f_{ij} f_{ji}^{-1} q^{-2(i-j)/n}, \quad i, j = 1, \dots, n, \quad (\text{A14})$$

$$\lambda_{ijst} = f_{ij} f_{st}^{-1} q^{-2(i-s)/n}, \quad i < s < j \text{ or } j < s < i \text{ and } t = i + j - s. \quad (\text{A15})$$

The q_{ij} and λ_{ijst} satisfy the following obvious symmetries,

$$q_{ji} = q_{ij}^{-1}, \quad i, j = 1, \dots, n, \quad (\text{A16})$$

$$\lambda_{jst} = q_{ji} \lambda_{ijst}, \quad i < s < j \text{ and } t = i + j - s, \quad (\text{A17})$$

$$\lambda_{ijts} = q_{st} \lambda_{ijst}, \quad i < s < j \text{ and } t = i + j - s. \quad (\text{A18})$$

Moreover, we see that modulo these symmetries every λ_{ijst} must either be of the form $\lambda_{ij\alpha\alpha}$, when $i + j$ is even, or $\lambda_{ij\alpha, \alpha+1}$ when $i + j$ is odd *or* be expressible in terms of these as

$$\lambda_{ijst} = \begin{cases} \lambda_{ij\alpha\alpha} / \lambda_{st\alpha\alpha} & i + j \text{ even,} \\ \lambda_{ij\alpha, \alpha+1} / \lambda_{st\alpha, \alpha+1} & i + j \text{ odd.} \end{cases} \quad (\text{A19})$$

Now, recalling the solution (A5), we find that that $q_{ij} = y^{-i+j} z^{i-j} q^{-2(i-j)/n}$, so that on defining $p = y^{-1} z q^{-2/n}$, we have that $q_{ij} = p^{i-j}$. Now consider the λ_{ijst} s. From (A5) we get

$$\begin{aligned} \lambda_{ij\alpha\alpha} &= x^{-(\alpha-i)^2} y^{(\alpha-i)(\alpha-i+1)} z^{(\alpha-i)(\alpha-i-1)} w^{-(\alpha-i)^2} q^{-2(i-\alpha)/n} \\ &= (y^{-1} z q^{-2/n})^{(i-\alpha)} (x^{-1} y z w^{-1})^{(\alpha-i)^2} \\ &= p^{(i-\alpha)} (x^{-1} y z w^{-1})^{(\alpha-i)^2}, \end{aligned} \quad (\text{A20})$$

$$\begin{aligned} \lambda_{ij\alpha, \alpha+1} &= x^{-(\alpha-i)(\alpha-i+1)} y^{(\alpha-i)(\alpha-i+2)} z^{(\alpha-i)^2} w^{-(\alpha-i)(\alpha-i+1)} q^{-2(i-\alpha)/n} \\ &= (y^{-1} z q^{-2/n})^{(i-\alpha)} (x^{-1} y z w^{-1})^{(\alpha-i)(\alpha-i+1)} \\ &= p^{(i-\alpha)} (x^{-1} y z w^{-1})^{(\alpha-i)(\alpha-i+1)}, \end{aligned} \quad (\text{A21})$$

so that on defining $\lambda = x^{-1} y z w^{-1}$, and recalling (A19), we find that

$$\lambda_{ijst} = p^{i-s} \lambda^{st-ij}, \quad i < s < j \text{ and } t = i + j - s. \quad (\text{A22})$$

This completes the derivation.

APPENDIX B. THE MULTIPARAMETER GENERALISED FRONSDAL-GALINDO R -MATRIX

In Proposition 3.4 we presented the 2-cocycle χ_{FG} defined on a certain standard quantum group in terms of a matrix F . To obtain the twisted quantum group we also need χ_{FG}^{-1} , which is defined in terms of F^{-1} ,

$$(F^{-1})_{ij}^{st} = \begin{cases} f_{ij}^{-1} & i = s, j = t, \\ \bar{\mu}_k & i = k, j = k', s = N, t = N, \\ \bar{\lambda}_{kl} & i = k, j = k', s = l, t = l', \end{cases} \quad (\text{B1})$$

where $0 < k < N$, $0 < k < l < N$, and,

$$\bar{\mu}_i = -q q^{i-i'} p_{ii'} f_{NN}^{-2} \mu_i, \quad 0 < i < N, \quad (\text{B2})$$

$$\bar{\lambda}_{ij} = -q^{2(i-j)} p_{ii'} p_{jj'} f_{NN}^{-2} \lambda_{ij}, \quad 0 < i < j < N. \quad (\text{B3})$$

Now we determine the twisted R -matrix, $R_{FG,p}$, from $R_{FG,p} = F_{21}R_{S,p}F^{-1}$ as

$$(R_{FG,p})_{ij}^{st} = \begin{cases} p_{ij}f_{ji}f_{ij}^{-1} & i = s, j = t, \\ \bar{\mu}_k f_{k'k} p_{kk'} & i = k, j = k', s = t = N, \\ \bar{\lambda}_{kl} f_{k'k} p_{kk'} & i = k, j = k', s = l, t = l', \\ \mu_k f_{NN}^{-1} p_{NN} & i = k', j = k, s = t = N, \\ \lambda_{kl} f_{ll}^{-1} p_{ll} & i = k', j = k, s = l', t = l, \\ (q - q^{-1}) & i = t < j = s, \end{cases} \quad (B4)$$

where $0 < k < N$ and $0 < k < l < N$, the p_{ij} s and f_{ij} s are constrained according to (67) and (69) respectively, and all other parameters are given in terms of the μ s. We can refine the presentation of this R -matrix, setting

$$\kappa_k = q^{-1} f_{NN} \bar{\mu}_k, \quad (B5)$$

$$\tilde{\kappa}_k = -q^{2(N-k)} \kappa_k, \quad (B6)$$

$$\xi_{kl} = (1 - q^2)(\kappa_k / \kappa_l), \quad (B7)$$

$$\tilde{\xi}_{kl} = (1 - q^{-2})q^{2(l-k)}(\kappa_k / \kappa_l). \quad (B8)$$

Then the R -matrix becomes the *multiparameter generalised Fronsda-Galindo R -matrix*,

$$(R_{FG,p})_{ij}^{st} = \begin{cases} q & i = j = s = t, \\ qp_{ii'}^2 & i = s = j', j = t, 0 < j < N, \\ q^{-1}p_{ii'}^2 & i = s, j = t = i', 0 < i < N, \\ p_{j'j} & i = s = N, j = t \neq N, \\ p_{ii'} & i = s \neq N, j = t = N, \\ p_{ij}p_{ii'}p_{j'i} & i = s \neq N, j = t \neq N, i \neq j, i + j \neq 2N, \\ q - q^{-1} & i = t < j = s, \\ qp_{ii'}\kappa_i & 0 < i < N, j = 2N - i, s = t = N, \\ qp_{j'j}\tilde{\kappa}_j & 0 < j < N, i = 2N - j, s = t = N, \\ q^{-1}p_{ii'}p_{ss'}\xi_{is} & 0 < i < s < N, j = 2N - i, t = 2N - s, \\ qp_{j'j}p_{tt'}\tilde{\xi}_{jt} & 0 < j < t < N, i = 2N - j, s = 2N - t. \end{cases} \quad (B9)$$

It is not difficult to see that, in general, this has $(1 + \frac{1}{2}(N-1)(N+4))$ parameters. However, owing to the particular p_{ij} s which appear in the R -matrix, in the cases of $N = 2$ and $N = 3$, the number of parameters is reduced to 3 and 7 respectively.

To identify the R -matrix originally discussed by Fronsda and Galindo (34), as a special case of this R -matrix, consider the particular solution of (67) given by setting $p_{ij} = 1$ for $0 < i \neq j < N$, $0 < i < N < j < 2N$ and $N < i \neq j < 2N$.

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